On Lorentzian Para-Sasakian Manifold admitting Semi- Symmetric metric connection

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Abstract

The aim of the present chapter is to find the scalar curvature of an LP-Sasakian manifold with respect to semi symmetric metric connection and proved that if Ricci tensor with respect to connection vanishes, then manifold is η- Einstein manifold. We have also deduced a necessary condition for the manifold becomes Einstein is that A (ξ)= 0.

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Introduction

Let (M^n ,g) be an n- dimensional Reimannian manifold of class C^∞ with metric g , A linear connection $\underline{\nabla}$ on (M^n ,g) is said to be semi - symmetric [1] , if the torsion tensor T of the connection satisfies

$$T(X,Y) = \pi(Y)X - \pi(X)Y \qquad (1.1)$$

Where π is a 1 – form on M with p as associated vector field, defined as

$$\pi (X) = g(X ,\rho) \qquad (1.2)$$

for any vector field X on M.

A semi - symmetric metric connection $\underline{\nabla}$ is called semi-symmectric metric connection [2] if it satisfies

$$\underline{\nabla}_x g = 0 \qquad (1.3)$$

Sharfuddin and Hussain [2] define a semi - symmetric metric connection in an almost contact manifold by identifying the 1- form π of (1,1) with the contact form η

$$T(X,Y) = \eta(Y) - \eta(X)Y \qquad (1.4)$$

In [3] U.C De and J. Sengupta studied a semi - symmetric metric connection on a almost contact manifold satisfying the conditions (1.4) and

$$(\underline{\nabla} xT)(Y,Z) = A(\varphi X)T(Y,Z) + A(X)\varphi(T(Y,Z)) \qquad (1.5)$$

Where A is 1- form and φ is a tensor field of type (1,1)

In this paper we have studied a semi - symmetric metric connection in LP-Sasakian manifold whose torsion tensor T, satisfies the conditions (1.4) and

(1.5). In section 2, the definition of LP- Sasakian manifold with some results are given. In section 3, we find the scalar curvature with respect to semi - symmetric metric connection.

2.Preliminaries

Let M be an n - dimensional C^{∞} manifold is called Lorentzian para Sasakian manifold [4] [5] if their exist in M a vector valued linear function φ , a contra - variant vector field ξ , and a covariant vector field η and a Lorentzian metric g , satisfying

$$\eta(\xi) = -1. \quad (2.1)$$

$$\phi^2 = I + \eta(X)\xi \quad (2.2)$$

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y) \quad (2.3)$$

$$g(X, \xi) = \eta(X) \quad (2.4)$$

$$(\nabla_x \varphi)(Y) = g(X,Y) + \eta(X)\eta(Y) + [X + \eta(X)\xi]\eta(Y) \quad (2.5)$$

where ∇ is the operator of covariant differentiation with respect to Lorentzian metric g .Also in a LP- Sasakian manifold the following relations holds

$$\Phi\xi = 0, \eta(\varphi X) = 0, rank = n - 1. \quad (2.6)$$

Also, an LP-Sasakian manifold M is said to be Einstein if its Ricci tensor S is of the form

$$S(X, Y) = \mu g(X, Y) + \upsilon\eta(X)\eta(Y). \quad (2.7)$$

For any vector fields X, Y where μ, υ are functions on M

In [6] K. Yano obtained the relation between the connection ∇ and the Levi - civita connection of space (M^n , g) and also find the relation between the curvature R and of type(1,3) as follow.

$$\underline{\nabla} xY = \nabla xY + \pi(Y)X - g(X,Y)\rho \quad (2.8)$$

And

$$\underline{R}(X, Y, Z) = R(X,Y)Z + \alpha(Y,Z)X - \alpha(Y,Z)LX - g(Y,Z)LX + g(X,Z)LY \quad (2.9)$$

Where $\alpha(Y,Z) = g(LY,Z)$

$$= (\nabla_y \pi)Z - \pi(Y)\pi(Z) + \frac{1}{2}\pi(\rho)g(Y,Z) \quad (2.10)$$

The Weyl conformal curvature tensor C of type (1 ,3) on the manifold is defined by [3]

$$C(X,Y) = g(QY,Z)$$
$$= -\frac{1}{n-1} S(Y,Z) + \frac{\lambda}{2(n-1)(n-2)} g(Y,Z)$$

Curvature Tensor of the semi - symmetric metric connection in LP-Sasakian manifolds

From (1.4) , we have

$$T(Y,Z) = \eta(Z)Y - \eta(Y)Z \quad (3.1)$$

Where $\eta(Z) = g(Z,\xi)$ (3.2)

Contracting (3.1) with respect to Y , we get

$$(C,T)(Z) = (n-1)\eta(Z),$$

where (C , T) denotes contraction of T

Taking covariant differentiation with respect to ∇ we get

$$(\underline{\nabla}_x, C, T)(Z) = (n-1)(\nabla_x\eta)(Z) \quad (3.3)$$

Also from (1.5) we have

$$\left(\underline{\nabla}_x T\right)(Y,Z) = A(X)T(Y,Z) + A(\Phi X)\Phi\big(T(Y,Z)\big) \quad (3.4)$$

Where A is a 1- form and Φ is a tensor field of type (1,1) .

Contracting (3.4) with respect to Y , we get

$$\left(\underline{\nabla}_x C,T\right)(Z) = A(X)(C,T)(Z) + A(\Phi X)(C,\Phi)(Y)\eta(Z) - \eta(\Phi Z)$$
$$= (n-1)A(X)\eta(Z) + aA(\Phi X)\eta(Z) \quad (3.5)$$
$$where \quad a = (C,\Phi)(Y) \quad and \quad \eta(\Phi Z) = 0$$

from (3.5) and (3.3) , we get

$$\left(\underline{\nabla}_x\eta\right)(Z) = A(X)\eta(Z) + bA(\varphi X)\eta(Z) \quad (3.6)$$

where $b = \frac{a}{n-1}$

Also from (3.2)

$$(\underline{\nabla}_x\eta)(Z) + \eta(\underline{\nabla}_x Z) = g(\nabla xZ,\xi) + g(Z,\nabla x\xi)$$
$$(\underline{\nabla}_x\eta)(Z) = g(Z,\underline{\nabla}_x\xi)$$

Using (2.8) we get

$$(\underline{\nabla}_x\eta)(Z) = g(Z,\nabla x\xi + \pi(\xi)X - \eta(X)p)$$
$$= g(Z,\nabla x\xi) + \pi(\xi)g(Z,X) - \eta(X)g(Z,\rho)$$

$$= (\nabla_x \eta)(Z) + \pi(\xi) g(Z,X) - \eta(X)\pi(Z) \qquad (3.7)$$

From (3.6) and (3.7)

$$\nabla_x \eta(Z) = A(X)\eta(Z) + bA(\Phi X)\eta(Z) - \pi(\xi) g(Z,X) + \eta(X)\pi(Z) \qquad (3.8)$$

From (2.10) if π = η and $\rho = \xi$, then

$$\alpha(X,Z) = A(X)\eta(Z) + bA(\varphi X)\eta(Z) + g(Z,X) + \frac{1}{2}(\varphi X) g(X,Z) \qquad (3.9)$$

$here\ \alpha(X,Z) \overset{\text{def}}{=} g(LX,Z)$

$And\ LX = A(X)\xi + bA(\varphi X)$ (3.10)

using (3.9) , (3.10) in (2.9) , we get

$$\underline{R}(X,Y,Z) = R(X,Y)Z + A(X)\eta(Z)Y + bA(\varphi X)\eta(Z)Y + g(Z,X)Y + \frac{1}{2}\eta(\rho)g(X,Z)Y - A(Y)\eta(Z)X - bA(\varphi X)\eta(Z)X - g(Z,Y)X - (1/2)\eta(\rho)g(Y,Z)X - g(Y,Z)A(X)\xi - g(Y,Z)bA(\varphi X)\xi - g(Y,Z)\xi - \frac{1}{2}\eta(\rho)g(Y,Z)X + g(X,Z)A(Y)\xi + g(X,Z)bA(\varphi Y)\xi + g(X,Z)Y + \frac{1}{2}\eta(\rho)g(X,Z)Y$$

$$\underline{R}(X,Y,Z) = R(X,Y)Z + A(X)Y - A(Y)X\eta(Z) + bA(\varphi X)Y - A(\varphi Y)X\eta(Z) + 2g(Z,X)Y - g(Z,Y)X + g(X,Z)Y - g(Y,Z)X\eta(\rho) + g(X,Z)A(Y)\xi - g(Y,Z)A(X) + bg(X,Z)A(\varphi Y)\xi - g(Y,Z)A(\varphi X)\xi \qquad (3.11)$$

Let $\underline{S}$ and S denote the Ricci tensor of the manifold with respect to $\underline{\nabla}$ and ∇ respectively

From (3.11) , we get

$$\underline{S}(Y,Z) = S(Y,Z) + (2-n)[A(Y)\eta(Z) + bA(\varphi Y)\eta(Z) + g(Y,Z)] + g(Y,Z)[(1-n)\eta(\rho) - A(\xi)]. \qquad (3.12)$$

Again from (3.12) , we get

$$\underline{r} = r + (1-n)[2A(\xi) + n\eta(p)] + n(2-n) \qquad (3.13)$$

Theorem (3.1) The scalar curvature with respect to semi - symmetric metric connection$\underline{\nabla}$ of an LP - Sasakian manifold admitting a semi symmetric metric connection is given by (3.13)

$$\underline{r} = r + (1-n)[2A(\xi) + n\eta(p)] + n(2-n)$$

Corollary (3.1) : The Ricci tensors S of the manifold with respect to semi symmetric metric connection in an LP – Sasakian manifold is symmetric if and only if

$$\eta(Z)[A(Y) + bA(\varphi Y)] = \eta(Y)[A(Z) + bA(\varphi Z)] \qquad (3.14)$$

In particular $\underline{S}$ = 0 Then from (3.12) , we get

$$S(Y,Z) = g(Y,Z)[A(\xi) - (n-1)\eta(\rho)] - (2-n)[A(Y) + bA(\varphi Y)]\eta(Z) + g(Y,Z). \qquad (3.15)$$

Putting Z = ξ in (3.14) and using in (3 .15)

$$S(Y,Z) = g(Y,Z)[A(\xi) - (n-1)\eta(\rho)] + (2-n)\eta(Y)\eta(Z)A(\xi) - (2-n)g(Y,Z) \qquad (3.16)$$

If $\underline{S}$ = 0 then $\underline{r}$ = 0 then from (3.13) we get

$$A(\xi) = \frac{1}{2(n-1)}[(r - n(n-1)\eta(\rho) - n(n-2)]$$

$$S(Y,Z) = \frac{r - (3n^2 - 5n + 2)\eta(\rho) + n^2 - 4n + 4}{2(n-1)} g(Y,Z) + \frac{2-n}{2(n-1)}[r - 2(n-1)^2\eta(\rho) - 2(n-1)(2-n)]\eta(Y)\eta(Z)$$

$$S(Y,Z) = \mu g(Y,Z) + \upsilon\eta(Y)n(Z)$$

$$Where\ \mu = \frac{r - (3n^2 - 5n + 2)\eta(\rho) + n^2 - 4n + 4}{2(n-1)}$$

And

$$\upsilon = \frac{2-n}{2(n-1)}[r - 2(n-1)^2\eta(\rho) - 2(n-1)(2-1)]$$

Which is η – Einstein manifold .

Theorem (3.2) If the Ricci tensor of an LP – Sasakian manifold with respect to the semi – symmetric metric connection vanishes , then the manifold reduces to η – Einstein manifold .

4. Weyl Conformal Curvature Tensor.

The Weyl conformal curvature tensor of type (1,3) in an LP – sasakian manifold with respect to connection is defined by

$$\underline{C}(X,Y,Z) = \underline{R}(X,Y,Z) + \underline{\lambda}(Y,Z)X - \underline{\lambda}(X,Z)Y + g(Y,Z)\underline{Q}X - g(X,Z)\underline{Q}Y \qquad (4.1)$$

where

$$\underline{\lambda}(Y,Z) = g(QY,Z) = \frac{1}{n-1}\underline{S}(Y,Z) + \frac{\underline{r}}{2(n-1)(n-2)}g(Y,Z) \qquad (4.2)$$

If in particular $\underline{S}= 0$, then $\underline{r} = 0$, Then from (4.2) , we get

$$\lambda(Y,Z) = 0 \qquad (4.3)$$

using this fact in (4.1), we get

$$\underline{C}(X,Y)Z = \underline{R}(X,Y)Z$$

Theorem (4.1) If the Ricci tensor of an LP – Sasakian manifold with respect to semi – symmetric metric connection vanishes , then Weyl conformal curvature tensor with respect to connection $\underline{\nabla}$ is equal to the curvature tensor of the manifold with respect to connection $\underline{\nabla}$

Again from (3.11) , we get

$$\begin{aligned}\underline{R}(X,Y)Z) + \underline{R}(Y,Z)X + \underline{R}(Z,X)Y &= \{A(Y) + bA(\varphi Y)\}\{\eta(X)Z - g(Z,X)\xi\} \\ &\quad -\{A(Z) + bA(\varphi Z)\}\{(\eta(X)Y - g(Y,X)\xi\} \\ &\quad +A(Z) + bA(\varphi Z)\eta(Y)X - g(X,Y)\xi \\ &\quad -\{A(X) + bA(\varphi X)\})\eta(Y)Z - g(Z,Y)\xi \\ &\quad + \{A(X) + bA(\varphi X)\})\{\eta(Z)Y - g(Y,Z)\xi) \\ &\quad - A(Y) + bA(\varphi Y)\eta(Z)X - g(X,Z)\xi \end{aligned} \qquad (4.4)$$

If we assume that $\underline{S}$ is symmectric Then (3.14) holds putting Z = ξ in (3.14), we get

$$A(Y) + bA(\varphi Y) = -\eta(Y)A(\xi) \qquad (4.5)$$

Using this result in (4.4) , we get

$$\underline{R}(X,Y)Z) + \underline{R}(Y,Z)X + \underline{R}(Z,X)Y = 0. \qquad (4.6)$$

Conversely, if (4.6) holds , them from (4.4)

$$[A(X) + bA(\varphi X)\eta(Z)Y - \eta(Y)Z] + [A(Y) + bA(\varphi Z)\eta(X)Z - \eta(Z)X] + [A(Z) + bA(\varphi Z)\eta(Y)X - \eta(X)Y] = 0 \qquad (4.7)$$

Contracting w. r. to X , we get

$$\eta\,(Y)A(Z) + bA\,(\varphi Z\,)\,) = \eta\,(Z)\{(A(Y) + bA\,(\varphi Y)\}$$

putting Z = ξ in above, we get

$$A(Y) + bA(\varphi Y) =- \eta(Y)A(\xi) \qquad (\,4.8\,)$$

We state the theorem:

Theorem (4.2) : The necessary and sufficient condition for the Ricci tensor of an LP - Sasakian manifold with respect to semi - symmetric metric connection $\underline{\nabla}$ to be symmetric is

$$\underline{R}(X,Y)Z) + \underline{R}(Y,Z)X + \underline{R}(Z,X)Y = 0$$

Now, if $\underline{R}$ = 0 then from (3.11), we get

$$S(Y,Z) = (2-n)\eta(Y)\eta(Z)A(\xi) - (2-n)g(Y,Z) - (1-n)g(Y,Z)\eta(\rho) + g(Y,Z)A(\xi)$$

$$S(Y,Z) = g(Y,Z)[n-2+\eta(\rho)(n-1)] + (2-n)\eta(Y)\eta(Z)A(\xi) + g(Y,Z)A(\xi) \qquad (\,4.9)$$

From (4.9), it follows that

$$S(Y,Z) = g(Y,Z)[n-2+\eta(\rho)(n-1)]$$

If A (ξ) = 0 Thus, we state the theorem:

Theorem (4.3) If the curvature tensor R of an LP - Sasakian manifold . with respect to the semi symmetric metric connection ∇ vanishes, then the necessary conditions for the manifold becomes Einstein manifold is A (ξ) = 0 .

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