APPLICATIONS OF DIFFERENTIAL EQUATIONS

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ABSTRACT : *In this paper we will discuss application of differential equations. Differential equations are generally used in many applied sciences. Mass spring system, free oscillations, undamaged system and damped system are studied. Problem has important role in the field of differential equations.*

1. INTRODUCTION : Differential equations are widely used in fields of Engineering and Applied Sciences. Mathematical formulations of most of the physical problems are in the forms of differential equations. Use of differential equations is most prominent in subjects like Circuit Analysis, Theory of Structures, Vibrations, Heat Transfer, Fluid Mechanics, etc. Differential equations are of two types : Ordinary and Partial. In ordinary equations, there is one dependent variable depending for its value on one independent variable. Partial differential equations will have more than one independent variables.

2. MASS-SPRING SYSTEM (VIBRATION OF SPRINGS)

LDE with constant coefficients play very important role in representing vibrating mechanical systems. In this section we discuss the motions of a basic mechanical system, attached to elastic spring (Fig.). Modeling of Mass-Spring system includes Corting up its mathematical equation, solving it and discussing the nature of motion.

2.1 MODEL OF MASS-SPRING SYSTEM (FREE OSCILLATIONS)

Let an ordinary spring (which resists compression as well as extension) be suspended vertically from a fixed support. At the lower end of the spring we attach a body of mass Im'. When the body is in rest, we describe this position as the equilibrium position. If we pull the body down a certain distance and then release it, it undergoes a motion. We shall determine the motion of mechanical system.

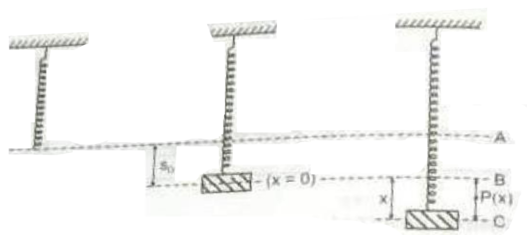
Spring restoring force : It has tendency to restore the system to its equilibrium position. It is governed by Hooke's law which states that "the force exerted by a spring, to restore the weight W to its equilibrium position is proportional to the distance of W from the equilibrium position (briefly restoring force is proportional to stretch). Thus if F is restoring force and x denote the position of W measured from the equilibrium position then

$$F \alpha x \text{ or } F = -kx$$

Here k (> 0) is the constant of proportionality, depends upon the stiffness of spring and is called spring constant and minus sign appears because force F points upward. We note here that a stiff spring has a large k: small stretch .

2.2 Model equation : Consider Fig. initially, the spring is unstretched (Fig.). When we attach the body of mass 'm' it stretches the spring by amount . This causes an upward force , in the spring. By Hooke's law this force , is proportional to the stretch So, thus

$$F_o = -kS_o \qquad \text{......(1)}$$



Unstretched spring (a) System in static equillibrium (b) System in motion (c)

The extension S_o is such that F_o balances the weight W of the body. Consequently,

$$F_0 + W = -k\,S_0 + mg = 0 \qquad \text{......(2)}$$

These forces will not affect the motion and the spring and the body again at rest is called the static equilibrium of the system (refer Fig. (b)). We take this position the body as origin (i.e. x = 0) and is used to measure the displacement x(t) of the body. From this position x = 0, we pull the body downwards which further stretches the sp by some amount x > 0 (the distance we pull down) (Fig. (c)). This cause additional upward restoring force F in the spring. By Hooke's law this force F_1 is proportional to stretch x. Thus,

$$F_1 = -kx \qquad \text{...... (3)}$$

Hence, F_1, is the only force causing motion for our mechanical system. This motion is governed by Newton's second law of motion, if x(t) is the displacement of the body and t is the time, then

$$m \times \frac{d^2x}{dt^2} = \text{Resultant force} \qquad ....(4)$$

3.0 UNDAMPED SYSTEM (FREE, UNDAMPED OSCILLATION)

Every system has damping otherwise it would keep moving forever. But here we consider those mechanical system with very ideal spring where effect of damping may eligible (e.g. the external forces such as air resistance and other forces) and con did not decrease. For instance, the motion of an iron ball on a spring during few minutes. Then F is the only force acting in (4), causing motion. Hence from (3) and (4)

$$m\frac{d^2x}{dt^2} + kx = 0 \quad or \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad ....(5)$$

Putting $\omega^2 = \frac{k}{m}$, the equation (5) takes the form

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

whose general solution is sinusoidal given by

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t \qquad ......(6)$$

Introducing $c_1 A\cos c_2 = A\sin\phi$, solution (6) can be rewritten as

$$x(t) = A\cos\phi + A\sin\phi \sin \omega t$$

or,

$$x(t) = A\cos(\omega t + \phi) \qquad .....(7)$$

where $A = \sqrt{c_1^2 + c_2^2}, \tan\phi = -\frac{c_2}{c_1}$. The constant A is called the amplitude of the motion and gives the maximum (positive) displacement of the mass from its equilibrium position. Thus the free, undamped motion of the mass is a simple harmonic motion, which is periodic. The period of the motion is the time interval between two successive maxima and is given by

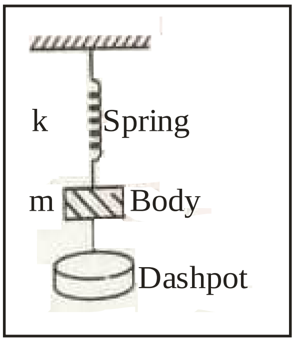
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}. \quad \text{....... (8)}$$

The natural frequency (or simply frequency) of the motion (or harmonic oscillation) is the reciprocal of the period, which gives the number of oscillations/second. Thus natural frequency is the undamped frequency i.e. frequency of the system without damping.

3.1 DAMPED SYSTEM (FREE, DAMPED OSCILLATION)

If the motion of the mass m be subjected to an additional force of resistance (damping or frictional forces of medium), the oscillations are said to be damped.

If we connect the dashpot (See in figure), we have to take corresponding viscous damping into account. The corresponding damping force has direction opposite to the instantaneous motion. We assume that it is proportional to the velocity $\frac{dx}{dt}$ of the body. (For small velocities this is a good approximation.) Thus the damping is of the form



$c(>0)$, constant of proportionality, is called the damping constant.

The resultant force acting on the body is now

$$F_1 + F_2 = -kx - c\frac{dx}{dt}$$

Hence, by Newton's second law, motion of the mass with damping force is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \ \ or \ \ \frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

4.0 MASS-SPRING SYSTEM (FORCED OSCILLATIONS)

Earlier we have discussed the problems of spring where only restoring and damping forces were working inspite of the weight W. We now consider cases where other external forces which depend on time may also act. Such forces may occur, for example, when the support holding the spring is moved up and down in a prescribed manner such as in periodic motion or when the weight is given a little push everytime it reaches the lowest position. If we denote the external force by F(t), the differential equation of motion is,

$$\frac{W}{g}\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + F(t)$$

or

$$\frac{W}{g}\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \qquad (W = mg)$$

This equation is called the equation of forced vibrations. In what follows, we shall now discuss the behaviour of mechanical system for two cases:

(i) damped forced oscillations (c>0) and (ii) undamped forced oscillations (resonance, c=0).

4.1 PROBLEM SECTION

1) *In the previous solved example, assume that a periodic external force given by* $F(g) = 24\cos\frac{14}{\sqrt{3}}t$ *is acting. Find x in terms of t, using conditions given there.*

Sol.: The differential equation will be

$$\frac{3}{9.8}\frac{d^2x}{dt^2} = -20x - 1.5\frac{dx}{dt} + 24\cos\frac{14}{\sqrt{3}}t$$

or

$$\frac{d^2x}{dt^2} + 4.9\frac{dx}{dt} + \frac{196}{3}x = 78.4\cos\frac{14}{\sqrt{3}}t \qquad ....(1)$$

with the initial conditions $x = \frac{1}{10}, \quad \frac{dx}{dt} = 0 \ \ at \ \ t = 0$ ....(2)

If we solve equation (1), the complementary function is given by,

$$C.F. = x_c = e^{-(2.45)t}[A\cos(7.7)t + B\sin(7.7)t]$$

and particular integral is

$$P.I = x_p = \frac{1}{D^2 + 4.9D + \frac{196}{3}} 78.4\cos\frac{14}{\sqrt{3}}t \Big| = 1.979\sin\frac{14}{\sqrt{3}}t$$

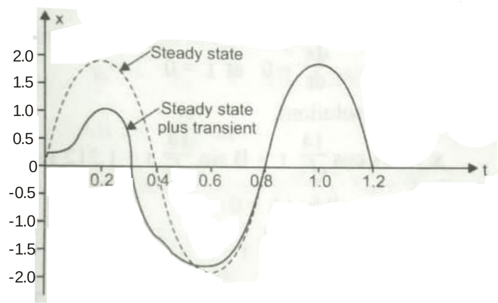
Hence the general solution of equation (1) is

$$x = x_c + x_p = e^{-(2.45)t}[A\cos(7.7)t + B\sin(7.7)t] + 19.979\sin\frac{14}{\sqrt{3}}t \qquad \text{.... (3)}$$

and using the initial conditions (2), we have

$$A = \frac{1}{10}, \qquad B = -2.046$$

and hence $x = e^{-(2.45)t}[1\cos(7.7)t - (2.046)\sin(7.7)t] + 1.979\sin\frac{14}{\sqrt{3}}t$ ....(4)



The graph of equation (3) is given in Fig. It will be seen that the terms in equation (4) involving $e^{-(2.45)t}$, become negligible (die down) when t is large. These terms are called Transient terms and are of value only when t is near zero, and these terms in the solution, when they are significant are called Transient solution. But when the transient terms are negligible, the term 1.979 sin $\frac{14}{\sqrt{3}}$ t remains. This is called the Steady state solution since it indicates the behaviour of the system when things have become steady. It is seen that steady state solution (shown dashed curve in figure) is periodic, having same period as that of applied external force.

5.0 MECHANICAL FORCE

When the frequency of a periodic external force, applied to a mechanical system is related to the natural frequency of the system, Mechanical resonance may occur, which builds up to oscillations to such tremendous magnitudes that the system itself may fall apart. A compancy of soldies marching in step across a bridge may in this manner cause the bridge to collapse even though it would have been strong enough to carry many more soldiers had they marched out of step. Similarly, it may be possible for a musical note of proper characteristic frequency to shatter a glass. Hence mechanical resonance in general should be avoided by engineers designing structure of vibrating mechanism. The example below will indicate what may be the consequence of resonance.

5.1 PROBLEM SECTION

1) *Suppose an external force given by* $6\cos\frac{14}{\sqrt{3}}t$ *is applied to the spring of the solved example 1 of art 3.3. Describe the motion which ensures if it is assumed that initially the weight is atthe equilibrium position (x=0) and that its initial velocity is zero.*

Sol.: The differential equation will be

$$\frac{3}{9.8}\frac{d^2x}{dt^2} = -20 + 6\cos\frac{14}{\sqrt{3}}t$$

or $$\frac{d^2x}{dt^2} + \frac{196}{3}x = 19.6\cos\frac{14}{\sqrt{3}}t \qquad .....(1)$$

and the initial conditions are

$$x = 0, \qquad \frac{dx}{dt} = 0 \ \ at \ \ t = 0 \qquad .....(2)$$

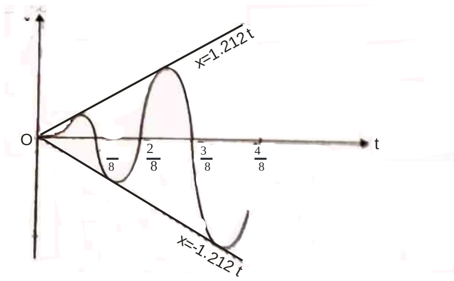
The solution will be (general solution)

$$x = A\cos\frac{14}{\sqrt{3}}t + B\sin\frac{14}{\sqrt{3}}t + 1.212t\sin\frac{14}{\sqrt{3}}t \qquad .....(3)$$

and using initial condition, it will be (A=0=B)

$$x = 1.212\sin\frac{14}{\sqrt{3}}t \qquad .....(4)$$

Graph of equation (4) will lie between the graphs of x=1.212t and x= -1.212t as shown in figure.



It is seen from the graph that oscillations build up without limit. Naturally the spring is bound to break within a short time.

It should be noted here that damping was neglected and *Resonance occurred because the frequency of the applied external force was equal to the Natural frequency of the undamped system.* This is a general principle. In the case where damping occurs, the oscillations do not build up without limit but may sometimes become large.

2) *A spring stretches 1 cm under the tension of 2N and has a negligible weight. It is fixed at one end and is attached to a weight W Newton at the other. It is found that resonance occurs when an axial periodic force 2 cos 2t N acts on the wight. Show that when the free vibrations have died out, the forced vibrations are given by x=ct sin 2t and find the values of W and c.*

Sol.: A weight of 2N stretches the spring by $\frac{1}{100}m$

$$\therefore \qquad 2 = T = k \cdot \frac{1}{100} \Rightarrow k = 200 N/m$$

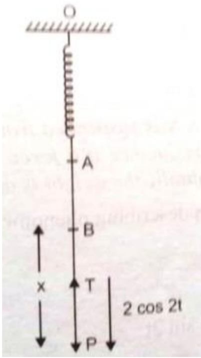
Let B be the equilibrium positions of the weight W attached to A, then

$$W = T_B = k \cdot AB = 200\,AB$$

$$\Rightarrow \qquad AB = \frac{W}{200} m$$

At any time t, let the weight be at P where BP=x. Then the tension T at P

$$= k \cdot AP = 200\left(\frac{W}{200} + x\right) = W + 200x$$



$\therefore$ Equation of motion is,

$$\frac{W}{g}\frac{d^2x}{dt^2} = -T + W + 2\cos 2t$$

$$\Rightarrow \quad \frac{W}{g}\frac{d^2x}{dt^2} = -W - 200 + W + 2\cos 2t$$

Or
$$\frac{d^2x}{dt^2} + 200\frac{gx}{W} = \frac{2g}{W}\cos 2t \quad ....(1)$$

The phenomenon of resonance will occur when the frequency of free oscillations is equal to the frequency of forced oscillations or (the period of free oscillations is equal to the period of forced oscillations).

If we write equation (1) as $\frac{d^2x}{dt^2} + \omega^2 x = \frac{2g}{W}\cos 2t$ where $\omega^2 = \frac{200g}{W}$, the period of free oscillation is found to be and the period of the forced oscillations is $2\pi/2 = \pi$.

Hence $\frac{2\mu}{\omega} = \pi \Rightarrow \omega = 2$ or $\frac{200g}{W} = \omega^2 = 4$

hence $W = 50g$

Taking this value of W in equation (1), we have

$$\frac{d^2X}{dt^2}+4x=\frac{1}{25}\cos 2t \qquad .....(2)$$

Now free oscillations are given by C.F. and forced oscillations by the P.I. Hence when the free oscillations have died out, the forced oscillations are given by the P.I. of (2)

Now P.I. equation (2) $=\frac{1}{25}\frac{1}{D^2+4}\cos 2t=\frac{1}{25}t\frac{1}{2D}\cdot\cos 2t$

$$=\frac{1}{100}t\sin 2t=ct\sin 2t$$

Hence $c=\frac{1}{100}$

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